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P. Hillion and A. Lakhtakia

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On an initial boundary value problem involving Beltrami–Moses fields in electromagnetic theory

By P. Hillion[1] and A. Lakhtakia[2]

[1] *Institut Henri Poincaré, 86 Bis Route de Croissy, 78110 Le Vésinet, France*
[2] *Department of Engineering Science and Mechanics, 227 Hammond Building, The Pennsylvania State University, University Park, PA 16802-1401, U.S.A.*

The so-called Harmuth ansatz consists of including autonomous magnetic sources in the time-dependent Maxwell postulates. The Beltrami fields are eigenfunctions of the curl operator, and have been used by Moses for propagation in infinite media.

These developments are of relatively recent provenances in electromagnetic theory. We discuss an initial-boundary value problem (IBVP) within the framework of a manifestly covariant electromagnetic formalism by using the Harmuth ansatz. We also show how a covariant formulation of the Beltrami–Moses fields may be used for solving electromagnetic IBVPs.

1. Introduction

Initial-boundary value problems (IBVPs) appear not to have been discussed carefully by the classical electromagnetic community. The standard electromagnetic IBVPs are usually explored with the dubious practice (Harmuth 1986; Barrett 1990) of ignoring the magnetic current density *ab initio* and using direct and inverse Fourier transforms. Harmuth (1986, 1990) and Barrett (1988) have given physical reasons for doubting the standard practice; from the mathematical point of view, however, the standard practice is dubious for the simple reason that boundary conditions may not be divorced from the initial conditions (Hillion 1990, 1991*a*). Instead, what is now called the Harmuth ansatz, and which has antecedents in early work on electromagnetic fields (e.g. Becker 1982), should be utilized: briefly, the Harmuth ansatz consists of including a magnetic current density in the time-dependent Maxwell postulates for analytical purposes, the magnetic current density being set equal to zero at the very end. We discuss here the Harmuth ansatz as it pertains to Beltrami fields.

2. Electromagnetic and Beltrami fields

(*a*) *Beltrami fields*

A Beltrami field $b(x)$ is an eigenfunction of the curl

$$\nabla \wedge \boldsymbol{b}(\boldsymbol{x}) = \lambda \boldsymbol{b}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^3, \tag{1}$$

the proportionality constant λ being an eigenvalue of the curl. In fluid mechanics, Beltrami fields are specific solutions of the three-dimensional Euler (Navier–Stokes) equations for incompressible flow without viscosity (with viscosity) such that the vorticity is parallel to the velocity. Such a situation is seen to occur in nature as well as in computer simulations of chaotic flows, and has been recently discussed by McLaughlin & Pironneau (1991).

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Beltrami fields also occur in static plasmas when the magnetic flux density aligns itself with the current density. This situation was intensively studied in the past (e.g. Van Kampen & Felderhof 1967), and has become the object of renewed study as well (Mett & Tataronis 1989; Salingros 1990; Yoshida 1991*a*). A (static) Beltrami magnetic field exerts no Lorentz force, for which reason it was introduced by Lust & Schlute (1954) to allow magnetic fields and large currents to exist simultaneously in stellar matter. The Lust–Schlute proposal was enthusiastically taken up by Chandrasekhar (1956, 1957); since then it has seen extensive use in astrophysics as well as magnetohydrodynamics, and reviews by Aly (1984) and by Zaghloul & Barajas (1990) are suggested for the interested reader. Parenthetically, it is noted that a proof of the impossibility (Parker 1958) of force-free magnetic fields has been shown to be false recently (Zaghloul 1989).

The emphasis in this communication being on electromagnetism, one notes at once that the form of the Maxwell postulates suggests the electromagnetic field can be defined in terms of Beltrami fields. Some 30 years ago, Rumsey (1961) had proposed a new way of solving the Maxwell postulates in free space: unbeknown to him; however, he had been anticipated considerably earlier by Silberstein (1907), and even by Fresnel (1822) in a manner of speaking. These proposals, as well as their progeny (Varadan *et al.* 1987) apply only to monochromatic fields giving rise to what is known as circular polarization (Chen 1983) in the microwave and the optics literatures. Moreover, these ideas have proven to be of great importance in studying natural optically active media (Lakhtakia 1991*a*, *b*) and bi-isotropic media (Sihvola & Lindell 1991; Lakhtakia & Diamond 1991).

(*b*) *Beltrami–Moses fields*

But the breakthrough for the use of the Beltrami fields in electromagnetism came from Moses (1971) who introduced a particular class of Beltrami fields with the following properties.

They are three-dimensional complex vectors $\boldsymbol{\chi}(\boldsymbol{x},\boldsymbol{p};\lambda)$ depending on $\boldsymbol{x}$ and on a vector $\boldsymbol{p}$ in the momentum space, and are eigenfunctions of the curl operator

$$\nabla\wedge\boldsymbol{\chi}(\boldsymbol{x},\boldsymbol{p};\lambda)=\lambda|\boldsymbol{p}|\,\boldsymbol{\chi}(\boldsymbol{x},\boldsymbol{p};\lambda), \tag{2}$$

with the three eigenvalues $\lambda=0,\pm1$. They satisfy the orthogonality and completeness relations

$$\int \mathrm{d}\boldsymbol{x}\,\boldsymbol{\chi}^{*}(\boldsymbol{x},\boldsymbol{p};\lambda)\cdot\boldsymbol{\chi}(\boldsymbol{x},\boldsymbol{p}';\mu)=\delta(\boldsymbol{p}-\boldsymbol{p}')\,\delta_{\lambda\mu}, \tag{3a}$$

$$\sum_{\lambda}\int \mathrm{d}\boldsymbol{p}\,\chi_j^{*}(\boldsymbol{x},\boldsymbol{p};\lambda)\,\chi_l(\boldsymbol{x}',\boldsymbol{p};\lambda)=\delta(\boldsymbol{x}-\boldsymbol{x}')\,\delta_{jl}, \tag{3b}$$

where χ_j denote the cartesian components of $\boldsymbol{\chi}$ and the asterisk denotes the complex conjugate. Moreover, one has the relations

$$\nabla\cdot\boldsymbol{\chi}(\boldsymbol{x},\boldsymbol{p};\lambda)=-\mathrm{i}(2\pi)^{-\frac{3}{2}}|\boldsymbol{p}|\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\,\delta_{\lambda0}, \tag{4a}$$

$$\chi^{*}(\boldsymbol{x},\boldsymbol{p};\lambda)=-\mathrm{e}^{-2\mathrm{i}\lambda\phi}\boldsymbol{\chi}(\boldsymbol{x},\boldsymbol{p};\lambda), \tag{4b}$$

where ϕ is the polar angle of $\boldsymbol{p}$.

Explicitly the Beltrami–Moses fields are given as

$$\boldsymbol{\chi}(\boldsymbol{x},\boldsymbol{r};0)=(2\pi)^{-\frac{3}{2}}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\hat{\boldsymbol{p}} \tag{5a}$$

for $\lambda = 0$, and

$$\boldsymbol{\chi}(\boldsymbol{x},\boldsymbol{p};\lambda) = -\frac{\lambda}{\sqrt{2}}(2\pi)^{-\frac{3}{2}}\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\begin{pmatrix}\dfrac{\hat{p}_1(\hat{p}_1+\mathrm{i}\lambda\hat{p}_2)}{1+\hat{p}_3}-1\\[2ex] \dfrac{\hat{p}_2(\hat{p}_1+\mathrm{i}\lambda\hat{p}_2)}{1+\hat{p}_3}-\mathrm{i}\lambda\\[2ex] \hat{p}_1+\mathrm{i}\lambda\hat{p}_2\end{pmatrix}, \tag{5b}$$

for $\lambda = \pm 1$, with $\hat{\boldsymbol{p}} = \boldsymbol{p}/|\boldsymbol{p}|$ and

$$\boldsymbol{p} = |\boldsymbol{p}|\,(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = |\boldsymbol{p}|\,(\hat{p}_1, \hat{p}_2, \hat{p}_3).$$

Recently Moses & Prosser (1990) have used the previous formalism to examine electromagnetic fields in an infinite medium of constant conductivity but they have not considered initial-boundary value problems.

(c) *Three-dimensional complex formalism for electromagnetism*

In the Heaviside–Lorentz conceptualization of electromagnetic fields, autonomous magnetic sources are not considered. However, there is no particular harm in introducing them into the Maxwell postulates, if only to obtain symmetric forms (Barrett 1990; Lakhtakia 1992). To begin with, we have the symmetrized Maxwell equations

$$\nabla\cdot\boldsymbol{E}(\boldsymbol{x},t) = \epsilon_0^{-1}\rho_{\mathrm{e}}(\boldsymbol{x},t), \quad \nabla\cdot\boldsymbol{H}(\boldsymbol{x},t) = -\mu_0^{-1}\rho_{\mathrm{m}}(\boldsymbol{x},t), \tag{6a, b}$$

$$\nabla\wedge\boldsymbol{E}(\boldsymbol{x},t) = -\mu_0\,\partial_t\boldsymbol{H}(\boldsymbol{x},t)+\boldsymbol{K}(\boldsymbol{x},t), \quad \nabla\wedge\boldsymbol{H}(\boldsymbol{x},t) = \epsilon_0\,\partial_t\boldsymbol{E}(\boldsymbol{x},t)+\boldsymbol{J}(\boldsymbol{x},t), \tag{6c, d}$$

where we have treated $\boldsymbol{E}, \boldsymbol{H}$ as the basic electromagnetic fields in free space; μ_0 and ϵ_0 are, respectively, the permeability and the permittivity of free space; while $\partial_t \cong \partial/\partial_t$. We let the electric current and charge densities be denoted by $\boldsymbol{J}$ and ρ_e, and the autonomous magnetic current and charge densities are denoted by $\boldsymbol{K}$ and ρ_m. The continuity equations

$$\nabla\cdot\boldsymbol{J}(\boldsymbol{x},t)+\partial_t\rho_e(\boldsymbol{x},t) = 0, \quad \nabla\cdot\boldsymbol{K}(\boldsymbol{x},t)+\partial_t\rho_m(\boldsymbol{x},t) = 0, \tag{7a, b}$$

are also to be noted.

Equations (6*a–d*) have been so stated as to be Lorentz-covariant. We note, however, that frequently in the electrical engineering literature (e.g. Harrington 1961), one has instead of (6*b*, *c*)

$$\nabla\wedge\boldsymbol{E} = -\mu_0\,\partial_t\boldsymbol{H}-\boldsymbol{K}, \quad \nabla\cdot\boldsymbol{H} = \mu_0^{-1}\rho_m;$$

but one cannot have Lorentz-covariant equations then.

Let us now use a formulation of electromagnetism covariant under the complex orthogonal group $O(3, C)$ isomorphic to the connected component of the Lorentz group. Such a manifestly covariant formulation has the distinct advantage of being more compact and, as we shall see, is suited well to Beltrami fields.

We start with the electromagnetic field tensor $F_{\mu\nu}$ $\mu,\nu = 0,1,2,3$, yielding the self-dual tensor (Corson 1954)

$$\tilde{F}_{\mu\nu} = F_{\mu\nu}+\tfrac{1}{2}\mathrm{i}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}, \quad \mathrm{i} = \sqrt{-1}, \tag{8}$$

where $\epsilon_{\mu\nu\alpha\beta}$ is the 4D-permutation tensor sometimes called the Levi-Civita anti-symmetrical tensor. $\tilde{F}_{\mu\nu}$ has three independent components forming a complex vector $\boldsymbol{Q}$ defined as (Hillion 1991*b*)

$$\boldsymbol{Q}(\boldsymbol{x},t) = \sqrt{\epsilon_0}\,\boldsymbol{E}(\boldsymbol{x},t)+\mathrm{i}\sqrt{\mu_0}\,\boldsymbol{H}(\boldsymbol{x},t). \tag{9}$$

After introducing the complex vector $\boldsymbol{W}$

$$\boldsymbol{W}(\boldsymbol{x},t) = \mathrm{i}\sqrt{\mu_0}\,\boldsymbol{J}(\boldsymbol{x},t)+\sqrt{\epsilon_0}\,\boldsymbol{K}(\boldsymbol{x},t), \tag{10}$$

and the complex charge scalar R

$$R(\boldsymbol{x}, t) = \epsilon_0^{-\frac{1}{2}}\rho_m(\boldsymbol{x}, t) + \mathrm{i}\mu_0^{-\frac{1}{2}}\rho_e(\boldsymbol{x}, t), \tag{11}$$

the Maxwell equations (6) take the simple form

$$\nabla \wedge \boldsymbol{Q}(\boldsymbol{x}, t) = \mathrm{i}(\epsilon_0 \mu_0)^{\frac{1}{2}} \partial_t \boldsymbol{Q}(\boldsymbol{x}, t) + \boldsymbol{W}(\boldsymbol{x}, t), \tag{12a}$$

$$\nabla \cdot \boldsymbol{Q}(\boldsymbol{x}, t) = -\mathrm{i}(\epsilon_0 \mu_0)^{-\frac{1}{2}} R(\boldsymbol{x}, t). \tag{12b}$$

One checks easily that equations (12) are covariant under the orthogonal group $O(3, C)$, and from these equations the continuity condition

$$\nabla \cdot \boldsymbol{W}(\boldsymbol{x}, t) + \partial_t R(\boldsymbol{x}, t) = 0 \tag{13}$$

is easily deduced.

We may now express the complex vectors $\boldsymbol{Q}$ and $\boldsymbol{W}$ in terms of the Beltrami–Moses fields by a Fourier-like transform

$$\boldsymbol{Q}(\boldsymbol{x}, t) = \sum_{\lambda} \int \boldsymbol{\chi}(\boldsymbol{x}, \boldsymbol{p}; \lambda)\, q(\boldsymbol{p}, t; \lambda)\, \mathrm{d}\boldsymbol{p}, \tag{14a}$$

$$\boldsymbol{W}(\boldsymbol{x}, t) = \sum_{\lambda} \int \boldsymbol{\chi}(\boldsymbol{x}, \boldsymbol{p}; \lambda)\, w(\boldsymbol{p}, t; \lambda)\, \mathrm{d}\boldsymbol{p}, \tag{14b}$$

the functions q and w being the Beltrami–Moses transforms of $\boldsymbol{Q}$ and $\boldsymbol{W}$. Similarly for the charge scalar R, we use the ordinary Fourier transform

$$R(\boldsymbol{x}, t) = (2\pi)^{-\frac{3}{2}} \int \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\, r(\boldsymbol{p}, t)\, \mathrm{d}\boldsymbol{p}. \tag{15}$$

Substituting (14), (15) into the Maxwell equations (12) yields ordinary differential equations (easier to solve) for the weight functions q, w, r. In particular, from (12a) and (14a, b) we get

$$\lambda|\boldsymbol{p}|\, q(\boldsymbol{p}, t; \lambda) = \mathrm{i}c_0^{-1} \partial_t\, q(\boldsymbol{p}, t; \lambda) + w(\boldsymbol{p}, t; \lambda), \tag{16}$$

where $c_0 = (\epsilon_0 \mu_0)^{-\frac{1}{2}}$. If w is assumed to be known, (16) has the solution

$$q(\boldsymbol{p}, t; \lambda) = \mathrm{i}c_0\, \mathrm{e}^{-\mathrm{i}\lambda c_0 |\boldsymbol{p}| t} \left\{ \int \mathrm{e}^{\mathrm{i}\lambda c_0 |\boldsymbol{p}| t} w(\boldsymbol{p}, t; \lambda)\, \mathrm{d}t + \text{const.} \right\}. \tag{17}$$

If we substitute (14a) and (15) into (12b), and take (4a) into account, we get

$$q(\boldsymbol{p}, t; 0) = (\mathrm{i}c_0/|p|)\, r(\boldsymbol{p}, t). \tag{18}$$

These relations, together with the complex conjugate ones, allow to solve any problem in a homogeneous infinite medium. As stated previously, Moses & Prosser (1991) have recently used these Beltrami fields; but because their formulation of electromagnetism is not manifestly covariant, they are led to more intricate expressions.

Remark 1. Because of the completeness and orthogonality conditions (3), the expansion coefficients q and w can be obtained from

$$q(\boldsymbol{p}, t; \lambda) = \int \boldsymbol{\chi}^*(\boldsymbol{x}, \boldsymbol{p}; \lambda; \lambda) \cdot \boldsymbol{Q}(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x},$$

$$w(\boldsymbol{p}, t; \lambda) = \int \boldsymbol{\chi}^*(\boldsymbol{x}, \boldsymbol{p}; \lambda) \cdot \boldsymbol{W}(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x}.$$

3. Harmuth's ansatz revisited

(*a*) *Harmuth's ansatz*

We begin with a homogeneous medium without electric and magnetic charges, and having a permeability scalar ϵ and a permittivity scalar μ (Harmuth 1990; Moses & Prosser 1990). Only currents carried by charged particles are considered (Becker 1982). The Maxwell equations satisfied under these considerations are

$$\nabla \wedge \boldsymbol{H} = \epsilon\,\partial_t \boldsymbol{E} + \boldsymbol{J}, \quad \nabla \wedge \boldsymbol{E} = -\mu\,\partial_t \boldsymbol{H} + \boldsymbol{K}, \tag{19a, b}$$

$$\nabla \cdot \boldsymbol{E} = 0, \quad \nabla \cdot \boldsymbol{H} = 0. \tag{19c, d}$$

The electric and the magnetic current densities, $\boldsymbol{J}$ and $\boldsymbol{K}$ respectively, are *intrinsic* to the medium; these current densities satisfy the equations

$$\boldsymbol{J} + \tau_{\mathrm{e}}\,\partial_t \boldsymbol{J} - \sigma \boldsymbol{E} = 0, \quad \boldsymbol{K} + \tau_{\mathrm{m}}\,\partial_t \boldsymbol{K} + s\boldsymbol{H} = 0, \tag{20a, b}$$

where σ is the electric conductivity, s is the magnetic conductivity, while τ_{e} and τ_{m} are two time constants. We will have ϵ, μ, σ, τ_{e}, τ_{m}, and s real and positive in the sequel.

Let us now write (19) and (20) in a manifestly covariant form as in §2*b*. This is possible only if we have

$$\tau_{\mathrm{e}} = \tau_{\mathrm{m}} = \tau, \quad \epsilon s = \mu\sigma. \tag{21a, b}$$

Then (20*a*, *b*) become the Harmuth–Hussain equations (1991) and we get

$$\tau = m\sigma/Ne^2 = m/\xi_\sigma, \tag{22}$$

where m is the particle mass and e is the particle charge, N is the number of charge carriers in a unit volume and ξ_σ is the Stokes friction coefficient.

Assuming that the conditions (21*a*, *b*) are fulfilled and using the complex fields of §2*c*, (19) and (20) become

$$\nabla \wedge \boldsymbol{Q}(\boldsymbol{x},t) = \mathrm{i}\eta\,\partial_t \boldsymbol{Q}(\boldsymbol{x},t) + \boldsymbol{W}(\boldsymbol{x},t), \quad \eta = (\epsilon\mu)^{\frac{1}{2}}, \tag{23a}$$

$$\boldsymbol{W}(\boldsymbol{x},t) + \tau\,\partial_t \boldsymbol{W}(\boldsymbol{x},t) = \mathrm{i}\alpha \boldsymbol{Q}(\boldsymbol{x},t), \quad \alpha = \sigma(\mu/\epsilon)^{\frac{1}{2}}, \tag{23b}$$

where $\boldsymbol{Q} = \surd\epsilon \boldsymbol{E} + \mathrm{i}\surd\mu \boldsymbol{H}$ and $\boldsymbol{W} = \surd\epsilon \boldsymbol{K} + \mathrm{i}\surd\mu \boldsymbol{J}$ now. Eliminating $\boldsymbol{W}$ or $\boldsymbol{Q}$ between these last two equations leads to

$$\mathrm{i}\eta\tau\,\partial_t^2 \begin{pmatrix}\boldsymbol{Q}\\ \boldsymbol{W}\end{pmatrix} + \mathrm{i}\eta\,\partial_t \begin{pmatrix}\boldsymbol{Q}\\ \boldsymbol{W}\end{pmatrix} - \tau\nabla \wedge \partial_t \begin{pmatrix}\boldsymbol{Q}\\ \boldsymbol{W}\end{pmatrix} + \mathrm{i}\alpha \begin{pmatrix}\boldsymbol{Q}\\ \boldsymbol{W}\end{pmatrix} - \nabla \wedge \begin{pmatrix}\boldsymbol{Q}\\ \boldsymbol{W}\end{pmatrix} = 0. \tag{24}$$

If we now assume the representations (14*a*, *b*) of $\boldsymbol{Q}$ and $\boldsymbol{W}$ in terms of the Beltrami–Moses fields, (23*a*, *b*) give for the weight functions the ordinary differential equations

$$\lambda|\boldsymbol{p}|\,q(\boldsymbol{p},t;\lambda) = \mathrm{i}\eta\,\partial_t q(\boldsymbol{p},t;\lambda) + w(\boldsymbol{p},t;\lambda), \tag{25a}$$

$$w(\boldsymbol{p},t;\lambda) + \tau\,\partial_t w(\boldsymbol{p},t;\lambda) = \mathrm{i}\alpha q(\boldsymbol{p},t;\lambda), \tag{25b}$$

whence

$$\mathrm{i}\eta\tau\,\partial_t^2 \begin{pmatrix}q\\ w\end{pmatrix} + (\mathrm{i}\eta - \lambda|\boldsymbol{p}|\,\tau)\,\partial_t \begin{pmatrix}q\\ w\end{pmatrix} + (\mathrm{i}\alpha - \lambda|\boldsymbol{p}|) \begin{pmatrix}q\\ w\end{pmatrix} = 0. \tag{26}$$

Let us remark that since $\rho_{\mathrm{e}} = \rho_{\mathrm{m}} = 0$ here, according to (18) $q(\boldsymbol{p},0;t) = 0$. Hence, we only need $\lambda = \pm 1$ here. It is also easy to prove that $w(p,t;0) = 0$; indeed from (25*b*) we get

$$w(\boldsymbol{p},t;0) = w(\boldsymbol{p})\,\mathrm{e}^{-t/\tau},$$

and substituting this last result into (25a) gives

$$q(\boldsymbol{p}, t; 0) = 0 = (\mathrm{i}/\eta)\, w(\boldsymbol{p})\, \tau\, \mathrm{e}^{-t/\tau},$$

so that $w(\boldsymbol{p}) = 0$.

(b) *Planar* TEM *waves*

Let us now consider planar TEM waves propagating in the z direction so that

$$E_z = H_z = Q_z = 0, \tag{27a}$$

$$H_x = -\sqrt{(\epsilon/\mu)}\, E_y, \quad H_y = \sqrt{(\epsilon/\mu)}\, E_x. \tag{27b}$$

Then, we deduce from (24)

$$\mathrm{i}\eta(1+\tau\,\partial_t)\,\partial_t\, Q_x + \mathrm{i}\alpha\, Q_x = -(1+\tau\,\partial_t)\,\partial_z\, Q_y, \tag{28a}$$

$$\mathrm{i}\eta(1+\tau\,\partial_t)\,\partial_t\, Q_y + \mathrm{i}\alpha\, Q_y = (1+\tau\,\partial_t)\,\partial_z\, Q_x, \tag{28b}$$

where $\partial_z = \partial/\partial_z$. On using the fields

$$Q_{\pm} = Q_x \pm \mathrm{i}Q_y. \tag{29}$$

Equations (28a, b) become

$$\eta\tau\,\partial_t^2\, Q_+ + \eta\,\partial_t\, Q_+ - (1+\tau\,\partial_t)\,\partial_z\, Q_+ + \alpha Q_+ = 0, \tag{30a}$$

$$\eta\tau\,\partial_t^2\, Q_- + \eta\,\partial_t\, Q_- + (1+\tau\,\partial_t)\,\partial_z\, Q_- + \alpha Q_- = 0. \tag{30b}$$

Of course, one has similar equations for W_x, W_y, W_+, W_-.

Let us now consider the Beltrami–Moses fields. Since $q(\boldsymbol{p}, t; 0) = 0$ already the condition $Q_z = 0$ implies $\chi_z(\cdot, \cdot, \pm 1) = 0$. Then we must have $p_1 = p_2 = 0$ as per (5_b); hence $\boldsymbol{\chi}(\cdot, \pm 1)$ becomes

$$\boldsymbol{\chi}(\boldsymbol{x}, \boldsymbol{p}; \pm 1) = \frac{1}{\sqrt{2}}(2\pi)^{-\frac{3}{2}}\mathrm{e}^{\mathrm{i}p_3 z}\begin{pmatrix}\pm 1\\ \mathrm{i}\\ 0\end{pmatrix}\delta(p_1)\,\delta(p_2), \tag{31}$$

for the present purposes. After using (14a) we get, therefore,

$$Q_x(z, t) = \frac{(2\pi)^{-\frac{3}{2}}}{\sqrt{2}}\int \mathrm{e}^{\mathrm{i}p_3 z}[q(p_3, t; 1) - q(p_3, t; -1)]\,\mathrm{d}p_3, \tag{32a}$$

$$Q_y(z, t) = \frac{\mathrm{i}(2\pi)^{-\frac{3}{2}}}{\sqrt{2}}\int \mathrm{e}^{\mathrm{i}p_3 z}[q(p_3, t; 1) + q(p_3, t; -1)]\,\mathrm{d}p_3, \tag{32b}$$

leading to

$$Q_+(z, t) = (4\pi^3)^{-\frac{1}{2}}\int \mathrm{e}^{\mathrm{i}p_3 z} q(p_3, t; -1)\,\mathrm{d}p_3, \tag{33a}$$

$$Q_-(z, t) = (4\pi^3)^{-\frac{1}{2}}\int \mathrm{e}^{\mathrm{i}p_3 z} q(p_3, t; 1)\,\mathrm{d}p_3. \tag{33b}$$

We note that $q(p_3, t; \pm 1)$ is a reduced notation for $q(p_3\hat{z}, t; \pm 1)$. Now, one has just to solve the differential equation (26) for obtaining $q(p_3, t; \lambda)$ and we get

$$q(p_3, t; \lambda) = A(p_3, \lambda)\,\mathrm{e}^{a_+(p_3, \lambda)t} + B(p_3, \lambda)\,\mathrm{e}^{a_-(p_3, \lambda)t}, \quad \lambda = \pm 1, \tag{34a}$$

with

$$2a_{\pm}(p_3, \lambda) = -\left(\frac{1}{\tau} + \frac{\mathrm{i}}{\eta}\lambda p_3\right) \pm \sqrt{\left[\left(\frac{1}{\tau} - \frac{\mathrm{i}}{\eta}\lambda p_3\right)^2 \frac{-4\alpha}{\tau}\right]}, \tag{34b}$$

while A and B are functions of p_3 and λ to be determined by the initial-boundary conditions. But because of the relations (27*b*) and according to the definition of the function $\boldsymbol{Q}$, one has

$$Q_+(x,t) = 0, \tag{35}$$

which requires, according to (33*a*) and (34*a*),

$$A(p_3, -1) = B(p_3, -1) = 0. \tag{35$'$}$$

Thus from now on, we concentrate on Q_- and rename it Q in the following sections.

Remark 2. Of course, one has similar results for W_x, W_y, W_+, W_-, with $w(p_3, t; \lambda)$ still given by (34*a*) but with some functions C and D that are different from A and B.

Remark 3. Q_+, Q_-, Q_z, are in fact the components ϕ_1^2, ϕ_2^1, ϕ_1^1 of a traceless second rank spinor $\phi_r^s (r, s = 1, 2; \phi_1^1 + \phi_2^2 = 0)$. It has been shown (Hillion 1991) that the spinor formalism of electromagnetism covariant under the group SL$(2, C)$ of the 2×2 unimodular matrices isomorphic to the connex component of the Lorentz group is a powerful tool to solve the problems of electromagnetic wave propagation. The Beltrami–Moses spinors are discussed elsewhere (Hillion 1992).

(*c*) *Harmuth's ansatz for* TEM *waves*

(i) *Statement of the problem*

So we consider the field $Q(z,t)$ that satisfies (30*b*) rewritten as

$$(\partial_t + 1/\tau)(\partial_t Q + c\,\partial_z Q) + (\alpha c/\tau)\,Q = 0, \quad c = n^{-1}, \tag{36}$$

in the domain $z \geqslant 0, t \geqslant 0$ with the initial boundary conditions

$$Q(0,t) = \surd 2\, h(t), t \geqslant 0; \quad Q(z,0) = \surd 2 f(z), z \geqslant 0; \quad \partial_t Q(z,0) = \surd 2\, g(z), z \geqslant 0, \tag{37}$$

where the factor $+\surd 2$ is put for convenience.

From (33*b*) and (34*b*), we obtain the Beltrami–Moses representation of the field $Q(z,t)$ as

$$Q(z,t) = (4\pi^3)^{-\frac{1}{2}} \int \mathrm{e}^{\mathrm{i}p_3 z} (A(p_3)\,\mathrm{e}^{a_+(p_3)\,t} + B(p_3)\,\mathrm{e}^{a_-(p_3)\,t})\,\mathrm{d}p_3, \tag{38}$$

with $$2a_\pm = -(1/\tau + \mathrm{i}cp_3) \pm ((1/\tau - \mathrm{i}cp_3)^2 - 4\alpha c/\tau)^{\frac{1}{2}}. \tag{38$'$}$$

Then, the initial boundary conditions become

$$h(t) = (2\pi)^{-\frac{3}{2}} \int [A(p_3)\,\mathrm{e}^{a_+(p_3)\,t} + B(p_3)\,\mathrm{e}^{a_-(p_3)\,t}]\,\mathrm{d}p_3, \tag{39a}$$

$$f(z) = (2\pi)^{-\frac{3}{2}} \int \mathrm{e}^{\mathrm{i}p_3 z} (A(p_3) + B(p_3))\,\mathrm{d}p_3, \tag{39b}$$

$$g(z) = (2\pi)^{-\frac{3}{2}} \int \mathrm{e}^{\mathrm{i}p_3 z} [a_+(p_3)\,A(p_3) + a_-(p_3)\,B(p_3)]\,\mathrm{d}p_3. \tag{39c}$$

Since one has three conditions for only two unknowns (i.e. $A(p_3)$ and $B(p_3)$) it is clear that the initial-boundary conditions cannot be chosen arbitrarily and one obtains easily the relations

$$h(0) = f(0), h'(0) = g(0), \tag{40a, b}$$

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and
$$n\tau h''(0)+nh'(0)-\tau g'(0)-f'(0)+\alpha h(0)=0, \tag{40c}$$
where the prime denotes differentiation with respect to the argument.

(ii) *First problem* ($\tau \mapsto \infty$)

It is interesting, first, to understand the nature of the difficulties to be encountered with the kind of initial-boundary value problem just defined and, secondly, to see how the Beltrami–Moses transform works. So we begin by considering the simple partial differential equation
$$\partial_t^2 Q(z,t)+c\,\partial_t\,\partial_z\,Q(z,t)=0, \tag{41}$$
obtained by letting $\tau \rightarrow \infty$ in (36).

Since the characteristics of (41) are the lines
$$z=\text{const.},\quad z-ct=\text{const.}, \tag{42}$$
the general solution of (41) must be
$$Q(z,t)=\phi(z)+\psi(z-ct), \tag{43}$$
where ϕ and ψ are arbitrary functions with continuous partial derivatives.

Let us now consider the initial boundary value problem with the data (37), that is,
$$Q(0,t)=\surd 2h(t), t\geqslant 0;\quad Q(z,0)=\surd 2f(z), z\geqslant 0;\quad \partial_t\,Q(z,0)=\surd 2g(z), z\geqslant 0, \tag{44}$$
satisfying the conditions
$$h(0)=f(0),\quad h'(0)=g(0),\quad h''(0)+cg'(0)=0. \tag{44'}$$
Taking (43) into account, it is easy to prove that the solution of the initial boundary value problem (41), (44) is
$$Q(z,t)=\surd 2\begin{cases} f(z)+\dfrac{1}{c}\displaystyle\int_{z-ct}^{z} g(\xi)\,\mathrm{d}\xi, & 0<ct<z,\\ h(t-z/c), & 0<z<ct. \end{cases} \tag{45}$$

Let us now use the Beltrami–Moses solution (38). For $\tau \mapsto \infty$, one deduces from (34*b*)
$$a_+=-\mathrm{i}p_3\,ct,\quad a_-=0, \tag{46}$$
so that the Beltrami–Moses solution (38) becomes
$$Q(z,t)=(4\pi^3)^{-\frac{1}{2}}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}p_3 z}[A(p_3)\,\mathrm{e}^{-\mathrm{i}p_3 ct}+B(p_3)]\,\mathrm{d}p_3, \tag{47}$$
with from the initial boundary conditions (44)
$$h(t)=(2\pi)^{-\frac{3}{2}}\int_{-\infty}^{+\infty}(A(p_3)\,\mathrm{e}^{-\mathrm{i}p_3 ct}+B(p_3))\,\mathrm{d}p_3, \tag{48a}$$
$$f(z)=(2\pi)^{-\frac{3}{2}}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}p_3 z}(A(p_3)+B(p_3))\,\mathrm{d}p_3, \tag{48b}$$
$$g(z)=(2\pi)^{-\frac{3}{2}}\int_{-\infty}^{+\infty}-\mathrm{i}p_3\,\mathrm{e}^{\mathrm{i}p_3 z}A(p_3)\,\mathrm{d}p_3. \tag{48c}$$

We deduce at once from (48*b*, *c*) for $t = 0$ and $z > 0$

$$A(p_3) = -(2\pi)^{-\frac{1}{2}} \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}p_3 z} \int^z \frac{1}{c} g(\xi)\, \mathrm{d}\xi, \tag{49a}$$

$$B(p_3) = (2\pi)^{-\frac{1}{2}} \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}p_3 z} \left(f(z) + \frac{1}{c} \int^z g(\xi)\, \mathrm{d}\xi \right). \tag{49b}$$

Substituting (49*a*, *b*) into (47) gives

$$\begin{aligned} Q(z,t) &= -\frac{1}{\pi\sqrt{2}} \int_0^\infty \mathrm{d}z' \int_{-\infty}^{+\infty} \mathrm{d}p_3\, \mathrm{e}^{\mathrm{i}p_3(z-ct-z')} \int^{z'} g(\xi)\, \mathrm{d}\xi \\ &\quad + \frac{1}{\pi\sqrt{2}} \int_0^\infty \mathrm{d}z' \int_{-\infty}^{+\infty} \mathrm{d}p_3\, \mathrm{e}^{\mathrm{i}p_3(z-z')} \left(f(z') + \frac{1}{c} \int^{z'} g(\xi)\, \mathrm{d}\xi \right) \\ &= -\sqrt{2} \int_0^\infty \mathrm{d}z'\, \delta(z' - (z-ct)) \frac{1}{c} \int^{z'} g(\xi)\, \mathrm{d}\xi \\ &\quad + \sqrt{2} \int_0^\infty \mathrm{d}z'\, \delta(z'-z) \left(f(z') + \frac{1}{c} \int^{z'} g(\xi)\, \mathrm{d}\xi \right). \end{aligned} \tag{50}$$

Now we have

$$-\int_0^\infty \mathrm{d}z'\, \delta(z' - (z-ct)) \int^{z'} g(\xi)\, \mathrm{d}\xi = \int_{z-ct} g(\xi)\, \mathrm{d}\xi \quad \text{for} \quad z - ct > 0, \tag{51a}$$

$$\int_0^\infty \mathrm{d}z'\, \delta(z'-z) \left(f(z') + \frac{1}{c} \int^{z'} g(\xi)\, \mathrm{d}\xi \right) = f(z) + \frac{1}{c} \int^z g(\xi)\, \mathrm{d}\xi \quad \text{for} \quad z > 0. \tag{51b}$$

Therefore, the substitution of (51*a*, *b*) into (50) yields the solution (45) in the domain $0 < ct < z$. If we substitute (49*a*, *b*) into the right-hand side of (48*a*), we get zero identically as we have yet not utilized the data $h(t)$.

From (48*a*) we get for $z = 0$ and $t > 0$

$$A(p_3) = (2\pi)^{-\frac{1}{2}} \int_0^\infty \mathrm{e}^{\mathrm{i}p_3 ct} h(t)\, \mathrm{d}t, \quad B(p_3) = 0, \tag{52}$$

by following a similar procedure. Then, substituting (51*a*, *b*) into (46) gives

$$\begin{aligned} Q(z,t) &= \frac{\sqrt{2}}{\pi} \int_0^\infty \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}p_3(z+ct'-ct)} h(t')\, \mathrm{d}p_3\, \mathrm{d}t', \\ &= \sqrt{2} \int_0^\infty \delta(ct' - (ct-z))\, h(t')\, \mathrm{d}t' = h(t - z/c) \quad \text{for} \quad ct - z > 0, \end{aligned} \tag{53}$$

which is the solution (45) in the interval $0 < z < ct$.

It is important to note the role played by the characteristic $z - ct = 0$ in this kind of initial boundary value problem. The solution takes two different forms in the domains $z - ct > 0$ and $z - ct < 0$ and is discontinuous across the characteristic. In passing, we also remark that we would have the same solution in the domain $z < 0$, $t < 0$ but there are no possible solutions in the domains $z > 0, t < 0$ and $z < 0, t > 0$.

(iii) *Second problem* ($\alpha = 0$)

Let us now look for the solutions of equation (36) assuming that $\alpha = 0$ so that (36) reduces to

$$(\partial_t + 1/\tau)(\partial_t Q(z,t) + c\,\partial_z Q(z,t)) = 0. \tag{54}$$

In terms of two arbitrary functions ϕ, ψ with continuous partial derivatives the general solution of (54) is given by

$$Q(z,t) = \mathrm{e}^{(z-ct)/c\tau} \int^{z} \mathrm{e}^{-\xi/c\tau}\, \phi(\xi)\,\mathrm{d}\xi + \psi(z-ct), \tag{55}$$

and reduces to (43) for $\tau \to \infty$. Using (55), one obtains

$$\begin{aligned} Q(z,t) &= \surd 2\,\mathrm{e}^{-t/\tau} f(z) + \frac{1}{c}\mathrm{e}^{(z/ct)/c\tau}\,\surd 2 \int_{z-ct}^{z} \mathrm{e}^{-\xi/c\tau}\left[g(\xi)+\frac{1}{\tau}f(\xi)\right]\mathrm{d}\xi, \quad 0<ct<z, \\ &= \surd 2\, h(t-z/c), \quad 0<z<ct, \end{aligned} \tag{56}$$

as solution of (54) for the initial boundary value problem with the data (37).

We now discuss, as for the case of $\tau \to \infty$, the Beltrami–Moses transform, with the $a_{\pm}(p_3)$ given by

$$a_{+}(p_3) = -\mathrm{i}cp_3, \quad a_{-}(p_3) = -\tau^{-1}, \tag{57}$$

as $\alpha = 0$. Taking (57) into account, the Beltrami–Moses transform becomes

$$Q(z,t) = (4\pi^3)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}p_3 z}(A(p_3)\,\mathrm{e}^{\mathrm{i}p_3 ct} + B(p_3)\,\mathrm{e}^{-t/\tau})\,\mathrm{d}p_3, \tag{58}$$

with the initial boundary data

$$h(t) = (2\pi)^{-\frac{3}{2}} \int_{-\infty}^{+\infty} (A(p_3)\,\mathrm{e}^{-\mathrm{i}p_3 ct} + B(p_3)\,\mathrm{e}^{-t/\tau})\,\mathrm{d}p_3, \quad t \geqslant 0, \tag{59a}$$

$$f(z) = (2\pi)^{-\frac{3}{2}} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}p_3 z}(A(p_3)+B(p_3))\,\mathrm{d}p_3, \quad z \geqslant 0, \tag{59b}$$

$$g(z) = -(2\pi)^{-\frac{3}{2}} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}p_3 z}\left(\mathrm{i}cp_3 A(p_3) + \frac{1}{\tau}B(p_3)\right)\mathrm{d}p_3, \quad z \geqslant 0. \tag{59c}$$

Assuming $B(p_3) = 0$, we deduce from (59a) that

$$A(p_3) = (2\pi)^{-\frac{1}{2}} \int_{0}^{\infty} \mathrm{e}^{\mathrm{i}cp_3 t} h(t)\,\mathrm{d}t, \tag{60}$$

and substituting (60) into (58) gives

$$\begin{aligned} Q(z,t) &= \frac{\surd 2}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}p_3\,\mathrm{e}^{\mathrm{i}p_3(z+ct'-ct)} \int_{0}^{\infty} h(t')\,\mathrm{d}t', \\ &= \surd 2 \int_{0}^{\infty} \delta(ct'-(ct-z))\,h(t')\,\mathrm{d}t' = \surd 2 h(t-z/c), \quad ct-z>0, \end{aligned} \tag{61}$$

which agrees with the solution (56). Now from (59b, c) we deduce

$$f(z) + \tau g(z) = (2\pi)^{-\frac{3}{2}} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}p_3 z}(1-\mathrm{i}cp_3\tau)\,A(p_3)\,\mathrm{d}p_3, \tag{62a}$$

$$c\tau f'(z) + \tau g(z) = -(2\pi)^{-\frac{3}{2}} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}p_3 z}(1-\mathrm{i}cp_3\tau)\,B(p_3)\,\mathrm{d}p_3, \tag{62b}$$

leading to
$$(1-\mathrm{i}cp_3\tau)A(p_3) = (2\pi)^{-\frac{1}{2}}\int_0^\infty \mathrm{e}^{-\mathrm{i}p_3 z}(f(z+\tau g(z))\,\mathrm{d}z, \tag{63a}$$

$$(1-\mathrm{i}cp_3\tau)B(p_3) = -(2\pi)^{-\frac{1}{2}}\int_0^\infty \mathrm{e}^{-\mathrm{i}p_3 z}(c\tau f'(z)+\tau g(z)\,\mathrm{d}z. \tag{63b}$$

But from (58) we have

$$Q(z,t)-c\tau\,\partial_z Q(z,t) = (4\pi^3)^{-\frac{1}{2}}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}p_3 z}[(1-\mathrm{i}c\tau p_3)A(p_3)\,\mathrm{e}^{-\mathrm{i}p_3 ct} + (1-\mathrm{i}c\tau p_3)B(p_3)\,\mathrm{e}^{-t/\tau}]\,\mathrm{d}p_3, \tag{64}$$

and substituting (63) into (64) gives

$$Q(z,t)-c\tau\,\partial_z Q(z,t) = \frac{\sqrt{2}}{2\pi}\int_0^\infty \mathrm{d}z'(f(z')+\tau g(z'))\int_{-\infty}^{+\infty}\mathrm{d}p_3\,\mathrm{e}^{\mathrm{i}p_3(z-ct-z')} - \frac{\sqrt{2}}{2\pi}\mathrm{e}^{-t/\tau}\int_0^\infty \mathrm{d}z'(c\tau f'(z')+\tau g(z'))\int_{-\infty}^{+\infty}\mathrm{d}p_3\,\mathrm{e}^{\mathrm{i}p_3(z-z')},$$

that is

$$Q(z,t)-c\tau\,\partial_z Q(z,t) = \sqrt{2}[f(z-ct)+\tau g(z-ct)-\tau\,\mathrm{e}^{-t/\tau}(cf'(z)+g(z))]. \tag{65}$$

The integration of this last differential equation is straightforward and one obtains complete agreement with the solution (56), as expected.

(iv) *The general case*

Let us now consider the general equation (36) with $\alpha \neq 0$. Since the function

$$\xi(z,t) = Q(z,t)-\mathrm{e}^{-\alpha z} \tag{66}$$

satisfies the partial differential equation (54), one obtains without any further calculations, but only by using (56) and (66), the solution of (36) for the initial-boundary data (37) as

$$Q(z,t) = \mathrm{e}^{-\alpha z}(1-\mathrm{e}^{-t/\tau}) + \sqrt{2}\,\mathrm{e}^{-t/\tau}f(z) + \frac{1}{c}\mathrm{e}^{(z-ct)/c\tau}\int_{z-ct}^{z}\mathrm{e}^{-\xi/c\tau}\left[\sqrt{2}g(\xi) + \frac{1}{\tau}(\sqrt{2}f(\xi)-\mathrm{e}^{-\alpha\xi})\right]\mathrm{d}\xi, \quad \text{for} \quad 0<ct<z, \tag{67a}$$

$$Q(z,t) = \mathrm{e}^{-\alpha z}-\delta(t-z/c)+\sqrt{2}h(t-z/c), \quad 0<z<ct. \tag{67b}$$

For $\alpha \neq 0$, the Beltrami–Moses transform is very intricate, but fortunately we just need the Beltrami–Moses transform of the function $S(z,t)$ discussed above.

4. Conclusion

Let us conclude by three remarks on the Beltrami–Moses fields, the solutions of the hyperbolic partial differential equation (36) and on the Harmuth ansatz.

1. As stated in §2*a*, the Beltrami–Moses fields were previously used in an infinite medium. On the other hand we are concerned in this paper with an initial-boundary value problem with data specified on $z=0, t=0$, and solution in $z>0, t>0$. Insofar as the two-dimensional space-time differential equation (36) is concerned, the

Beltrami–Moses transform does not appear as powerful as expected. This may be for two reasons.

(i) Because (36) is a very simple partial differential equation, it is easy to guess that its general solution depends on two functions $\phi(z)$ and $\psi(z-ct)$. One can easily match the sum of ϕ and ψ to the prescribed initial-boundary data.

(ii) Except when $\alpha = 0$ and the Beltrami–Moses transform reduces to a Fourier transform, it is not transparently obvious how to obtain the functions $A(p_3)$ and $B(p_3)$ of (38) from the initial-boundary data.

Of course, the Beltrami–Moses transform continues to be of interest for the partial differential equations whose general solutions are difficult to guess.

2. The data for the initial-boundary value problem discussed here is given on a non space-like surface (the normal to a space-like surface is everywhere inside the light cone), which feature leads to great mathematical difficulties (Courant & Hilbert 1972, p. 754) concerning the existence and the properties of the solutions. Fortunately, the two-dimensional space-time hyperbolic equation is simple enough: available are the conditions (here the relations (40)) that the initial-boundary data must satisfy in order that a solution exists.

This solution has the important property of being discontinuous with the discontinuity propagating along the characteristic $z-ct = 0$, which is a well-known result (Courant & Hilbert 1962). Above the characteristic, the solution is the 'retarded' excitation $h(t-z/c)$ in agreement with the fact that no field can travel with a velocity greater than c. Below the characteristic the solution depends on the boundary data $f(z)$ and $g(z)$.

3. The final point concerns the autonomous magnetic charge and current densities and the Harmuth ansatz. We have noted in §2*c* that $\boldsymbol{K}$ and ρ_m are necessary for a Lorentz-covariant formulation of electromagnetic fields, and are also strongly suggested by the chiral invariance of the Maxwell postulates (Zwanziger 1968; Tiwari 1990). Whereas a non-zero $\partial_t \rho_m$ implies a $\boldsymbol{K}$ whose divergence is not zero, there can exist purely solenoidal $\boldsymbol{K}$ (i.e. $\nabla \boldsymbol{K} = 0$) which cannot be connected to time-varying magnetic charges densities.

It is the divergenceless $\boldsymbol{K}$ (and $\boldsymbol{J}$) that are investigated under the Harmuth ansatz in §3. This is quite clear from the defining equation (20*b*), particularly if we bear in mind that the propagation of planar TEM waves is considered. In a more general medium, for which $\tau_e = \tau_m = \tau$, but $s/\sigma \neq \mu/\epsilon$, the temporal variation of the electric field will give rise to a transverse magnetic field, and vice versa. Therefore, in that case, we must have both the electric and the magnetic fields associated with the excited planar TEM wave to be identically zero for $t < 0$ for the IBVP to be solved in $z > 0, t > 0$.

But (19*a*–*d*) and (20*a*, *b*) can be written in a manifestly covariant form if and only if (21*a*) and (21*b*) hold; this case is precisely what has been investigated in §3 and has constituted the theme of this paper. The scalar field Q_- (called simply Q from §3*c* onwards) contains a definite proportion of electric and magnetic fields, fixed for all time and everywhere. Hence, to have $Q_-(z=0,t) = 0$ for $t < 0$ is not necessary; indeed, such a condition is an unwarranted overspecification in light of the previous paragraph.

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